Exercise 5

Use residues to derive the integration formulas in Exercises 1 through 6.

$$\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}.$$

Solution

The integrand is an even function of x, so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \int_{-\infty}^\infty \frac{x^2\,dx}{2(x^2+1)(x^2+4)}$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{z^2}{2(z^2+1)(z^2+4)},$$

and the contour in Fig. 99. Singularities occur where the denominator is equal to zero.

$$\begin{gathered} 2(z^2+1)(z^2+4) = 0 \\ z^2+1 = 0 \quad \text{or} \quad z^2+4 = 0 \\ z = \pm i \quad \text{or} \quad z = \pm 2i \end{gathered}$$

The singular points of interest to us are the ones that lie within the closed contour, $z = i$ and $z = 2i$.

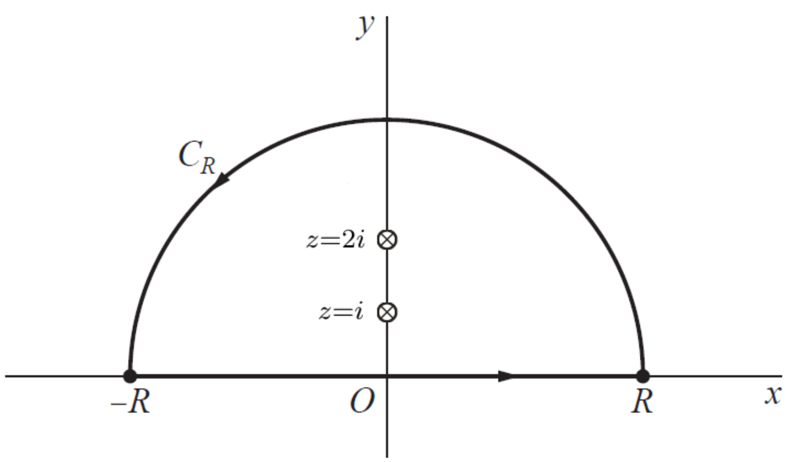

Figure 1: This is Fig. 99 with the singularities at $z = i$ and $z = 2i$ marked.

According to Cauchy's residue theorem, the integral of $z^2/[2(z^2+1)(z^2+4)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^2\,dz}{2(z^2+1)(z^2+4)} = 2\pi i \left[\operatorname*{Res}_{z=i} \frac{z^2}{2(z^2+1)(z^2+4)} + \operatorname*{Res}_{z=2i} \frac{z^2}{2(z^2+1)(z^2+4)}\right]$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{z^2\,dz}{2(z^2+1)(z^2+4)} + \int_{C_R} \frac{z^2\,dz}{2(z^2+1)(z^2+4)} = 2\pi i\left[\operatorname*{Res}_{z=i} \frac{z^2}{2(z^2+1)(z^2+4)} + \operatorname*{Res}_{z=2i} \frac{z^2}{2(z^2+1)(z^2+4)}\right]$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z = r, & r &= -R \quad \to \quad r = R \\ C_R: &\quad z = Re^{i\theta}, & \theta &= 0 \quad \to \quad \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{r^2\,dr}{2(r^2+1)(r^2+4)} + \int_{C_R} \frac{z^2\,dz}{2(z^2+1)(z^2+4)} = 2\pi i\left[\operatorname*{Res}_{z=i} \frac{z^2}{2(z^2+1)(z^2+4)} + \operatorname*{Res}_{z=2i} \frac{z^2}{2(z^2+1)(z^2+4)}\right].$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{r^2\,dr}{2(r^2+1)(r^2+4)} = 2\pi i\left[\operatorname*{Res}_{z=i} \frac{z^2}{2(z^2+1)(z^2+4)} + \operatorname*{Res}_{z=2i} \frac{z^2}{2(z^2+1)(z^2+4)}\right]$$

The denominator can be written as $2(z^2+1)(z^2+4) = 2(z+i)(z-i)(z+2i)(z-2i)$. From this we see that the multiplicities of the $z-i$ and $z-2i$ factors are both 1. The residues at $z = i$ and $z = 2i$ can then be calculated by

$$\begin{aligned} \operatorname*{Res}_{z=i} \frac{z^2}{2(z^2+1)(z^2+4)} &= \phi_1(i) \\ \operatorname*{Res}_{z=2i} \frac{z^2}{2(z^2+1)(z^2+4)} &= \phi_2(2i), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are equal to $f(z)$ without the $z-i$ and $z-2i$ factors, respectively.

$$\begin{aligned} \phi_1(z) &= \frac{z^2}{2(z+i)(z+2i)(z-2i)} \quad \Rightarrow \quad \phi_1(i) = \frac{i}{12} \\ \phi_2(z) &= \frac{z^2}{2(z+i)(z-i)(z+2i)} \quad \Rightarrow \quad \phi_2(2i) = -\frac{i}{6} \end{aligned}$$

So then

$$\begin{aligned} \operatorname*{Res}_{z=i} \frac{z^2}{2(z^2+1)(z^2+4)} &= \frac{i}{12} \\ \operatorname*{Res}_{z=2i} \frac{z^2}{2(z^2+1)(z^2+4)} &= -\frac{i}{6} \end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{r^2\,dr}{2(r^2+1)(r^2+4)} &= 2\pi i\left(\frac{i}{12}-\frac{i}{6}\right)\\ &= 2\pi i\left(-\frac{i}{12}\right)\\ &= \frac{\pi}{6}.\end{aligned}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}.}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 99 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{z^2\,dz}{2(z^2+1)(z^2+4)} &= \int_0^\pi \frac{(Re^{i\theta})^2(Rie^{i\theta}\,d\theta)}{2[(Re^{i\theta})^2+1][(Re^{i\theta})^2+4]} \\ &= \int_0^\pi \frac{R^3ie^{i3\theta}}{(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+4)}\frac{d\theta}{2}\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{z^2\,dz}{2(z^2+1)(z^2+4)}\right| &= \left|\int_0^\pi \frac{R^3ie^{i3\theta}}{(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+4)}\frac{d\theta}{2}\right| \\ &\leq \int_0^\pi \left|\frac{R^3ie^{i3\theta}}{(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+4)}\right|\frac{d\theta}{2} \\ &= \int_0^\pi \frac{\left|R^3ie^{i3\theta}\right|}{|R^2e^{i2\theta}+1|\,|R^2e^{i2\theta}+4|}\frac{d\theta}{2} \\ &= \int_0^\pi \frac{R^3}{|R^2e^{i2\theta}+1|\,|R^2e^{i2\theta}+4|}\frac{d\theta}{2} \\ &\leq \int_0^\pi \frac{R^3}{(|R^2e^{i2\theta}|-|1|)(|R^2e^{i2\theta}|-|4|)}\frac{d\theta}{2} \\ &= \int_0^\pi \frac{R^3}{(R^2-1)(R^2-4)}\frac{d\theta}{2} \\ &= \frac{\pi}{2}\frac{R^3}{(R^2-1)(R^2-4)}\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{C_R} \frac{z^2\,dz}{2(z^2+1)(z^2+4)}\right| &\leq \lim_{R\to\infty}\frac{\pi}{2}\frac{R^3}{(R^2-1)(R^2-4)} \\ &= \lim_{R\to\infty}\frac{\pi}{2R}\frac{1}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)}\end{aligned}$$

The limit on the right side is zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^2\,dz}{2(z^2+1)(z^2+4)}\right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^2\,dz}{2(z^2+1)(z^2+4)}\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{z^2\,dz}{2(z^2+1)(z^2+4)} = 0.$$